Another conection between φ and Fibonacci sequence

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August 10, 2020

Abstract

A map equation based on the φ number shows three different destinations. Two of them correspond to the fixed points of the system. The third is an indetermination of type 1/0. Interestingly, the route to the third destination is related to the Fibonacci sequence.

1 The Phi-bonacci map

The golden ratio or golden number φ is the positive solution of the following equation:

$$x = \frac{1}{x-1} \tag{1}$$

However, we can transform this equation into a map:

$$x_{n+1} = \frac{1}{x_n - 1} \tag{2}$$

This map has two fixed points $X_1^* = -0.618033988...$ and $X_2^* = \varphi$ that correspond to the two solutions of the polynomial equation $x^2 - x - 1 = 0$.

However the map in Eq. 2 has also an asymptote at $x_n = 1$ where the indetermination $\frac{1}{0}$ occurs. If we follow the map inversely to find a route that leads to the indetermination, we find:

$$x_n = 1 = \frac{1}{x_{n-1} - 1} \therefore x_{n-1} = \frac{2}{1}$$

$$x_{n-1} = \frac{2}{1} = \frac{1}{x_{n-2} - 1} \therefore x_{n-2} = \frac{3}{2}$$

$$x_{n-3} = \frac{5}{3},\ x_{n-4} = \frac{8}{5},\ x_{n-5} = \frac{13}{8}$$

This is the same as solving the inverse map:

$$x_{n+1} = \frac{1}{x_n} + 1,\ x_n = 1 \tag{3}$$

In this way we find that all the terms x_{n-i} can be expressed as

$$x_{n-i} = \frac{f_{n+1}}{f_n}, \tag{4}$$

where f_{n+1} and f_n are successive terms of the Fibonacci sequence.

The way to indetermination is paved with the numbers on Fibonacci sequence.